CSCI 2244 – Homework 5

Out: Friday, September 27, 2019
Due: Friday, October 4, 2019, 11:59pm

This homework consists of **only written** exercises. You *must* type your solutions. See the "Assignments" section in the syllabus for advice about doing this. You should submit your homework via Canvas. In particular, you should upload a PDF file called:

`FirstName_LastName_Homework5.pdf`

Please use your full first name and last name, as they appear in official university records. The reason for doing so is that the TAs and I must match up these names with the entries in the gradebook.

1 Written Exercises

1.1 More Practice with Conditional Probability and Expectations

Task 1.1 (2 pts). You are presented with two boxes. Both boxes contain two biased coins. The coins in box one both have probability p_1 of being heads, while the coins in box two have probability p_2 of being heads. You do not know the values of p_1 or p_2, but you are told that $p_1 \neq p_2$. You get to play a game where you can either:

(a) open one of the boxes randomly, and flip both coins it contains, or

(b) open both boxes, take one coin from each, and then flip the two coins.

You win the game if both coins you flip turn up heads. Show that option (a) is a better strategy than option (b). (Hint: Let $P(A)$ and $P(B)$ be the probabilities of winning with options a and b respectively. Show that $P(A) - P(B)$ can be written in the form $\frac{1}{2}(p_1 - p_2)^2$.)

Task 1.2 (1 pts). Suppose we roll a fair six-sided die. Let X be the random variable giving the value shown on the die. What is $\mathbb{E}[X^2]$?

Task 1.3 (1 pts). Let X be a random variable such that $P(X = -1) = .5$ and $P(X = 1) = .5$. What are $\mathbb{E}[X]$ and $\mathbb{E}[X^2]$?

Task 1.4 (3 pts). Suppose a waiter collects a credit card from each person in a party of n people at a restaurant. He piles the n cards up, charges each of them, and then hands back a random card from the pile to each person in the party. Let X be the number of people who get back the correct credit card. What is $\mathbb{E}[X]$? (Hint: Let X_i be an indicator variable which is 1 if the ith person got the right credit card. How can you write X in terms of the X_i?)

1.2 Poisson Distribution

Task 1.5 (4 pts). Let X be a Poisson random variable with parameter λ. Show that for all $k \in \mathbb{N}$:

$$P(X = k+1) = \frac{\lambda}{k+1} P(X = k)$$

Use this to prove that if the parameter λ is a positive integer, then $P(X = k)$ is largest when $k = \lambda$ and $k = \lambda - 1$. (That is, λ and $\lambda - 1$ are equally likely, and are the most likely values for X to take on).

Task 1.6 (3 pts). A company uses 1000 computer servers to process customer records. There are 10000 customer records. Each record is sent to one of the 1000 servers randomly, with each server equally likely. You want to understand whether some servers will be more overloaded than others. Use a Poisson approximation to estimate the probability that none of the servers receives more than 20 records – that is, all servers receive ≤ 20 records). (Hint: You can use the fact that if X is $\mathsf{Binomial}(10000, .001)$, then $P(X \leq 20) = 0.998421$).

Task 1.7 (3 pts). Use a Poisson approximation to estimate the probability that in a room of 40 people, there are at least 2 different days on which at least 2 people in the room have been born. (For example, if there are 3 people born on January 1st, and 2 born on September 27th, that counts.) As usual, assume there are no leap years and that the distribution of birthdays is uniform over the year.

You might consider writing Monte Carlo simulations to check if your approximations seem right, but you do not have to.